MATHEMATICS

STABILIZATION CONTROLLABILITY AND OBSERVABILITY OF LINEAR AUTONOMOUS SYSTEMS

BY

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1. *Introduction*

In this paper A, B, H denote real matrices of dimensions $n \times n$, $n \times m$, $r \times n$, respectively.

Consider the continuous control system

$$(\mathscr{L}_c): \qquad \dot{x}(t) = Ax(t) + Bu(t), \;\; y(t) = Hx(t),$$

for $t \in T_c := (0, \infty)$. The functions u, x, y are defined on T_c and have values in R^m, R^n, R^r, respectively. The function u is called a *control* or *input variable* and it is called *admissible* if it is integrable on each finite interval. The set of admissible controls is denoted by Ω_c.

Consider also the discrete control system

$$(\mathscr{L}_d): \qquad x(t+1) = Ax(t) + Bu(t), \;\; y(t) = Hx(t),$$

for $t \in T_d := \{0, 1, 2, \dots\}$. Here the functions u, x, y are defined on T_d and are vector valued of the same dimensions as in the continuous case. The set of *admissible controls* (or *input variables*) is denoted by Ω_d and consists of all sequences $u = \{u(0), u(1), \dots\}$.

In order to avoid duplication we make the following convention: If a definition or a theorem is understood to apply for both the continuous and the discrete case, we replace the indices c or d by an asterisk.

The function x is called the *state variable* and y the *output variable*. For every $u \in \Omega_*$, $a \in R^n$ the solution of $(\mathscr{L}_*)$ corresponding to u with initial value $x(0) = a$ is denoted by $x_u(t, a)$.

Definition 1. The system $(\mathscr{L}_*)$ is called *controllable* if for every $a, b \in R^n$, there exists $u \in \Omega_*$, $t \in T_*$ such that $x_u(t, a) = b$. The pair (A, B) is called *controllable* if the $n \times nm$-matrix $[B, AB, \dots, A^{n-1}B]$ has rank n. An eigenvalue λ of A is called (A, B)-*controllable* (or shorter *controllable* if there is no danger of confusion) if rank $[A - \lambda I, B] = n$. Equivalently: λ is controllable if there does not exist a row vector $\eta \neq 0$ such that $\eta A = \lambda \eta$, $\eta B = 0$.

Now we have the following result:

Theorem 1. The following facts are equivalent:

i) $(\mathscr{L}_*)$ is controllable,
ii) (A, B) is controllable,
iii) Every eigenvalue of A is controllable.

The equivalence of i) and ii) is well known ([4] p. 81, [2] p. 170). In [1] it is shown that ii) and iii) are equivalent.

Definition 2. $(\mathscr{L}_*)$ is called *null-controllable* if for every $a \in R^n$ there exists $u \in \Omega_*$, $t \in T_*$ such that $x_u(t, a) = 0$. $(\mathscr{L}_*)$ is called *asymptotically controllable*, if for every $a \in R^n$ there exists $u \in \Omega_*$ such that $x_u(t, a) \to 0$ $(t \to \infty)$.

It is known that $(\mathscr{L}_c)$ is null-controllable if and only if it is controllable ([4] p. 84, [3] p. 40). On the other hand it is possible that $(\mathscr{L}_d)$ is null-controllable without being controllable. (For instance, if A is nilpotent (that is, $A^k = 0$ for some k) and $B = 0$.) Also it is possible that $(\mathscr{L}_*)$ is asymptotically controllable without being controllable. (For example, if A is stable (see Def. 3) and $B = 0$.) Conditions for null-controllability of $(\mathscr{L}_d)$ and asymptotic controllability of $(\mathscr{L}_*)$ will be given in section 3.

Definition 3. The set of eigenvalues of A is called the *spectrum* of A and is denoted by $\sigma(A)$. The *characteristic polynomial* of A is denoted by χ_A and is defined by $\chi_A(z) := \det(zI - A)$. An eigenvalue λ of A is called (c)-*stable* if $\operatorname{Re} \lambda < 0$, and (d)-stable if $|\lambda| < 1$. Eigenvalues of A which are not $(\star)$-stable are called $(\star)$-*unstable*. The matrix A is called $(\star)$-*stable* if all eigenvalues of A are $(\star)$-stable.

It is well known that, if $u = 0$, the system $(\mathscr{L}_*)$ is asymptotically stable (that is, $x(t) \to 0$ $(t \to \infty)$ for all solutions of $(\mathscr{L}_*)$) if and only if A is $(\star)$-stable.

Definition 4. The system $(\mathscr{L}_*)$ is called *stabilizable* if there exists an $m \times n$-matrix D such that $A + BD$ is $(\star)$-stable.

Stabilizability is of importance for the synthesis of feedback controls. A control is called a feedback if it is described as a function of the state variable x, that is, $u = g(x)$. If a system is stabilizable, there exists a linear feedback $u = Dx$, which reduces $(\mathscr{L}_*)$ to a linear, autonomous, homogeneous, asymptotically stable system.

If a system is controllable, it is stabilizable. In fact, we have the much stronger result:

Theorem 2. (A, B) is controllable if and only if for every real polynomial $p(z)$ of degree n with coefficient of z^n equal to unity, there exists a real $m \times n$-matrix D such that $p = \chi_{A+BD}$. This statement is still true if the word real is omitted wherever it occurs.

For the case $m = 1$ this result is well known and easily proved by transforming the pair (A, B) into $(\bar{A}, \bar{B})$, where $\bar{A}$ is a companion matrix and $B = (0, \ldots, 0, 1)'$ (here the prime denotes transposition) (see [3] p. 49,

[4] p. 97). Theorem 2 is proved in [7] by means of a generalized companion matrix. In section 2 we will give a different proof, which depends on a combinatorial theorem of RADO ([6], [5] p. 537).

It is very well possible that $(\mathscr{L}_*)$ is stabilizable without being controllable (for example, if A is $(\star)$-stable and $B=0$). A necessary and sufficient condition for the stabilizability of $(\mathscr{L}_*)$ will be given in section 3. It is based on the following general result (proved in section 2):

Theorem 3. If S is a nonempty set of complex numbers, then there exists a matrix D with $\sigma(A+BD) \subset S$ if and only if every $\lambda \in \sigma(A)\backslash S$ is controllable. If $S \cap \bar{S} \neq \emptyset$ (where $\bar{S} := \{\bar{s} | s \in S\}$), then D can be chosen real.

In section 4 we will turn our attention to the observation of systems.

Definition 5. The system $(\mathscr{L}_*)$ is called *observable* if for all $u \in \Omega_*$ we have: $Hx_u(t, a) = Hx_u(t, b)$ $(t \in T_*)$ implies $a=b$ (and hence $x_u(t, a) = x_u(t, b)$ $(t \in T_*)$). The system is called *asymptotically observable* if $Hx_u(t, a) = Hx_u(t, b)$ $(t \in T_*)$ implies $x_u(t, a) - x_u(t, b) \to 0$ $(t \to \infty)$. The pair (A, H) is called *observable* if (A', H') is controllable. An eigenvalue λ of A is called (A, H)-*observable* (or *observable*) if it is (A', H')-controllable, hence if there exists no column vector $c \neq 0$ with $Ac = \lambda c$, $Hc = 0$.

It is well known that $(\mathscr{L}_*)$ is observable if and only if (A, H) is observable ([2] p. 170, [4] p. 111–112). It follows from Theorem 1 that (A, H) is observable if and only if every eigenvalue of A is observable.

Definition 6. An *asymptotic state estimator* for $(\mathscr{L}_c)$ is a system (with u, y as input and $\tilde{x}$ as output), of the form

$$(\mathscr{E}_c) \qquad \dot{z} = Pz + Qy + Ru, \quad \tilde{x} = Kz,$$

where z, y, u, $\tilde{x}$ are vector-valued functions of dimensions $\tilde{n}$, r, m, n and where P, Q, R, K are matrices of corresponding dimensions, such that for every $u \in \Omega_c$ and every $a \in R^n$, $b \in R^{\tilde{n}}$, we have

$$x_u(t, a) - Kz_{u,y}(t, b) \to 0 \qquad (t \to \infty),$$

where $y(t) = Hx_u(t, a)$. An asymptotic state estimator for $(\mathscr{L}_d)$ is defined similarly.

We will give necessary and sufficient conditions for the asymptotic observability and the existence of an asymptotic state estimator in section 4.

Sometimes it is desirable to stabilize $(\mathscr{L}_*)$ by a feedback which depends on the output y (instead of on the state x). In general this cannot be done by means of a feedback of the form $u = Dy$. (For instance, if $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $H = (1, 0)$, then the system $(\mathscr{L}_c)$ is controllable and observable, but $A + BDH$ has a (c)-unstable eigenvalue for every 1×1-matrix D.) Therefore, we need a different kind of stabilization:

Definition 7. The system $(\mathscr{L}_c)$ is called *indirectly (output-)stabilizable* if there exists a system $(\mathscr{S}_c)$ (with input y and output u) of the form:

$$(\mathscr{S}_c): \qquad \dot{z} = Pz + Qy, \ u = Dz$$

such that the composite system $(\mathscr{L}_c)$, $(\mathscr{S}_c)$:

$$\begin{aligned}\dot{x} &= Ax + BDz \\ \dot{z} &= Pz + QHx\end{aligned}$$

is asymptotically stable. A similar definition applies to $(\mathscr{L}_d)$.

Remark. Contrary to $(\mathscr{S}_*)$ a stabilization of the form $u = Dy$ is sometimes called a *direct stabilization.*

In section 4 we will give necessary and sufficient conditions for the indirect stabilizability of $(\mathscr{L}_*)$.

In section 5 we will give an application to sampled systems.

2. *Spectrum assignment*

In this section we will prove Theorems 2 and 3. First, we make the following observation:

Lemma 1. If $\lambda \in \sigma(A)$ is not (A, B)-controllable, then $\lambda \in \sigma(A + BD)$ for every $m \times n$-matrix D.

In fact, in that case there exists a row vector $\eta \neq 0$ with $\eta A = \lambda \eta$, $\eta B = 0$, and hence $\eta(A + BD) = \lambda \eta$ for every D.

Proof of Theorem 2. The sufficiency is a direct consequence of Lemma 1 and Theorem 1. The proof of the necessity consists in reducing the general problem to the special case $m = 1$. The theorem of Rado referred to in the introduction is the following one:

Theorem (Rado). If $\Sigma = \{S_1, \ldots, S_n\}$ is a collection of subsets of a vector space V such that for $k = 1, \ldots, n$ the union of each k-tuple of sets in Σ contains at least k independent vectors, then there exists a set of independent vectors $\{x_1, \ldots, x_n\}$ in V such that $x_k \in S_k$ $(k = 1, \ldots, n)$. For a proof see [6].

We need some further lemmas:

Lemma 2. If (A, B) is controllable and $P_k := [B, AB, \ldots, A^{k-1}B]$ $(k = 1, \ldots, n)$, then rank $P_k \geqslant k$ $(k = 1, \ldots, n)$.

Proof. The inequality is obvious for $k = 1$. If for some $k > 1$ we have rank $P_k < k$, there exists $\nu < k$ with rank $P_\nu =$ rank $P_{\nu+1}$. (Note that rank P_i depends increasingly on i.) This means that the columns of $A^\nu B$ are linear combinations of the columns of P_ν. But then the columns of $A^{\nu+1}B$ are linear combinations of the columns of AP_ν and hence of the columns of $P_{\nu+1}$. Therefore we have rank $P_{\nu+2} =$ rank $P_{\nu+1}$. By induction it follows that rank $P_n =$ rank $P_\nu \leqslant$ rank $P_k < k \leqslant n$, which contradicts the controllability of (A, B).

Lemma 3. Let (A, B) be controllable and let $M = \{\mu_1, \ldots, \mu_k\}$ denote a set of k distinct numbers, with $1 \leqslant k \leqslant n$ and $M \cap \sigma(A) = \emptyset$. Then, if $Q := [(A - \mu_1 I)^{-1}B, \ldots, (A - \mu_k I)^{-1}B]$, we have rank $Q \geqslant k$.

Proof. Let $\varphi(z) := (z - \mu_1) \ldots (z - \mu_k)$ and $\varphi_\nu(z) := \varphi(z)/(z - \mu_\nu)$ $(\nu = 1, \ldots, k)$. Then $\varphi(A)$ is nonsingular and hence rank $Q =$ rank $\varphi(A)Q =$ rank $[\varphi_1(A)B, \ldots, \varphi_k(A)B]$. The polynomials φ_ν have degree $k-1$ and are linearly independent. Therefore, we may write $z^p = \sum_{\nu=1}^{k} \alpha_{\nu p}\varphi_\nu(z)$ $(p = 1, \ldots, k)$, and hence $A^pB = \sum_{\nu=1}^{k} \alpha_{\nu p}\varphi_\nu(A)B$ $(p = 1, \ldots, k)$. This implies that rank $P_k \leqslant$ $\leqslant$ rank $[\varphi_1(A)B, \ldots, \varphi_k(A)B]$, and the result follows from Lemma 2.

Lemma 4. If (A, B) is controllable and $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ denotes a set of n distinct real numbers such that $\Lambda \cap \sigma(A) = \emptyset$, there exists a real $m \times n$-matrix D such that $\Lambda = \sigma(A + BD)$.

Proof. Let $S_p := \{(A - \lambda_p I)^{-1}Bu \mid u \in R^m\}$ $(p = 1, \ldots, n)$. It follows from Lemma 3 that each k-tuple of S_p's contains at least k independent vectors. It follows from Rado's theorem that there exists a basis $\{x_1, \ldots, x_n\}$ of R^n such that x_p is of the form $x_p = (A - \lambda_p I)^{-1}Bu_p$ with $u_p \in R^m$ $(p = 1, \ldots, n)$. Now we define D by $Dx_p = -u_p$ $(p = 1, \ldots, n)$. Then we have $(A - \lambda_p I)x_p =$ $= -BDx_p$, and hence $(A + BD)x_p = \lambda_p x_p$ $(p = 1, \ldots, n)$. This completes the proof of Lemma 4.

We are now in a position to prove the necessity part of Theorem 2. Suppose that (A, B) is controllable. First we choose a set Λ which satisfies the conditions of Lemma 4. Let D_1 denote a matrix with $\sigma(A + BD_1) = \Lambda$. It is easily seen from Theorem 1, iii) that $(A + BD_1, B)$ is also controllable (see also [1] example 1). Since there corresponds one independent eigenvector to each $\lambda \in \sigma(A + BD_1)$, it follows from [1] theorem 2, that there exists $c \in R^n$ such that $(A + BD_1, Bc)$ is controllable. Since for $m = 1$ the theorem is known we can find a row vector d such that $p = \chi_{A+BD_1+Bcd}$. Hence $D := D_1 + cd$ is the required matrix.

Proof of Theorem 3. Suppose that each $\lambda \in \sigma(A) \backslash S$ is controllable. A transformation of state space $\bar{x} = Tx$ transforms (A, B) into $(\bar{A}, \bar{B}) :=$ $= (T^{-1}AT, T^{-1}B)$ and it is easily seen that (A, B)-controllable eigenvalues are also $(\bar{A}, \bar{B})$-controllable. By the canonical decomposition theorem ([4] p. 99) we can choose T such that we have the following block-partitions:

$$\bar{A} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},$$

where A_{11} is a $p \times p$-matrix and B a $p \times m$-matrix, such that (A_{11}, B_1) is controllable. We have $\sigma(A) = \sigma(\bar{A}) = \sigma(A_{11}) \cup \sigma(A_{22})$. If $\lambda \in \sigma(A_{22})$, then λ is not $(\bar{A}, \bar{B})$-controllable and hence $\lambda \in S$. Therefore we have $\sigma(A_{22}) \subset S$. Furthermore, since (A_{11}, B_1) is controllable, it follows from Theorem 2 that there exists D_1 with $\sigma(A_{11} + B_1D_1) \subset S$. If $S \cap \hat{S} \neq \emptyset$, then D_1 can be chosen real. With $\bar{D} := [D_1, 0]$ we have $\sigma(\bar{A}+\bar{B}\bar{D}) = \sigma(A_{11}+B_1D_1) \cup \cup\, \sigma(A_{22}) \subset S$. Finally, with $D = \bar{D}T^{-1}$ we obtain $\sigma(A+BD) = \sigma(\bar{A}+\bar{B}\bar{D}) \subset S$.

The necessity part of the theorem is an easy consequence of Lemma 1.

3. *Controllability and direct stabilizability*

We have the following result:

Theorem 4. The following statements are equivalent:

i) $(\mathscr{L}_*)$ is asymptotically controllable.
ii) Every $(\star)$-unstable eigenvalue of A is controllable.
iii) $(\mathscr{L}_*)$ is stabilizable.

Proof. Suppose that $(\mathscr{L}_*) = (\mathscr{L}_c)$.

i) $\Rightarrow$ ii): If for some $\lambda \in \sigma(A)$ with $\operatorname{Re} \lambda \geqslant 0$ there exists a row vector $\eta \neq 0$ with $\eta A = \lambda\eta$, $\eta B = 0$ and if $\eta a \neq 0$, then we have $(d/dt)(\eta x_u(t, a)) = = \lambda\eta x_u(t, a)$. Hence, $\eta x_u(t, a) = e^{\lambda t}\eta a \not\to 0$ $(t \to \infty)$ for every $u \in \Omega_c$. Therefore, $(\mathscr{L}_c)$ is not asymptotically controllable.

ii) $\Rightarrow$ iii): This is an immediate consequence of Theorem 3.

iii) $\Rightarrow$ i): Let D stabilize $(\mathscr{L}_c)$ and let the solution of $\dot{x} = (A+BD)x$ with $x(0) = a$ be denoted by $\hat{x}(t, a)$. Then, with $u(t) := D\hat{x}(t, a)$ we have $x_u(t, a) = \hat{x}(t, a)$ $(t > 0)$, and hence $x_u(t, a) \to 0$ $(t \to \infty)$.

Similar reasoning applies to $(\mathscr{L}_*) = (\mathscr{L}_d)$.

Remark. It follows from this theorem in particular, that a system which can be stabilized by an arbitrary feedback $u = g(x)$, can also be stabilized by a linear feedback $u = Dx$.

Theorem 5. The following propositions are equivalent:

i) $(\mathscr{L}_d)$ is null-controllable.
ii) Every nonzero eigenvalue of A is controllable.
iii) There exists a null-control of $(\mathscr{L}_d)$ in the form of a linear feed-back.

The proof is analogous to the one of Theorem 4. Note that iii) is equivalent to "There exists D such that $\sigma(A+BD) = \{0\}$".

4. *Observability and indirect stabilizability*

Theorem 6. The following propositions are equivalent:

i) There exists an asymptotic state estimator for $(\mathscr{L}_*)$.
ii) The system $(\mathscr{L}_*)$ is asymptotically observable.
iii) Every $(\star)$-unstable eigenvalue of A is observable.

Proof for the case $(\mathscr{L}_*) = (\mathscr{L}_c)$:

i) $\Rightarrow$ ii): Let $(\mathscr{E}_c)$ be an asymptotic state estimator for $(\mathscr{L}_c)$ and suppose that $Hx_u(t, a) = Hx_u(t, b)$ $(t \geqslant 0)$. Then we have $x_u(t, a) - Kz_{u,y}(t, 0) \to 0$, and $x_u(t, b) - Kz_{u,y}(t, 0) \to 0$, where $y(t) = Hx_u(t, a) = Hx_u(t, b)$. Hence, $x_u(t, a) - x_u(t, b) \to 0$ $(t \to \infty)$.

ii) $\Rightarrow$ iii): If $(\mathscr{L}_c)$ is asymptotically observable and if for some $\lambda \in \sigma(A)$ there exists $c \neq 0$ with $Ac = \lambda c$, $Hc = 0$, then we have $(d/dt)x_0(t, c) = Ax_0(t, c)$, and hence $x_0(t, c) = e^{tA}c = e^{t\lambda}c$. Here $x_0(t, c)$ is the solution of $(\mathscr{L}_c)$ corresponding to the control $u = 0$. It follows that we have $Hx_0(t, c) = Hx_0(t, 0) = 0$ $(t \geqslant 0)$. Hence, $x_0(t, c) - x_0(t, 0) = e^{t\lambda}c \to 0$ $(t \to \infty)$. Therefore we have $\operatorname{Re} \lambda < 0$.

iii) $\Rightarrow$ i): Suppose that every (c)-unstable eigenvalue is observable. By Theorem 3 there exists a matrix L such that $A + LH$ is (c)-stable. It follows that, if $\tilde{n} = n$, $P = A + LH$, $Q = -L$, $R = B$, $K = I$ in $(\mathscr{E}_c)$, and if we define $v := \tilde{x} - x$, then we have $\dot{v} = (A + LH)\, v$, and hence $\tilde{x}(t) - x(t) \to 0$.

Asymptotic state estimators of the type given in the proof of iii) $\Rightarrow$ i), are discussed in [3] p. 55–57.

Theorem 7. The system $(\mathscr{L}_*)$ is indirectly output-stabilizable if and only if every $(\star)$-unstable eigenvalue is controllable and observable.

Proof for the case $(\mathscr{L}_*) = (\mathscr{L}_c)$:

Suppose that $(\mathscr{L}_c)$ is indirectly stabilizable by the system $(\mathscr{S}_c)$. Consider the matrix

$$\hat{A} := \begin{bmatrix} A & BD \\ QH & P \end{bmatrix}.$$

If $\lambda \in \sigma(A)$ is (c)-unstable and not controllable, then $\eta A = \lambda \eta$, $\eta B = 0$ for some $\eta \neq 0$. But then we have $\hat{\eta}\hat{A} = \lambda \hat{\eta}$, where $\hat{\eta} := (\eta, 0)$. Hence $\lambda \in \sigma(\hat{A})$. A similar argument applies to (c)-unstable eigenvalues which are not observable.

On the other hand suppose that every (c)-unstable eigenvalue is controllable and observable. According to Theorem 3 there exist matrices D and L such that $A + BD$ and $A + LH$ are (c)-stable. Consider the indirect feedback:

$$(\mathscr{S}_c'): \qquad \dot{z} = (A + LH + BD)\, z - LHx, \qquad u = Dz.$$

The coefficient matrix of the composite system $(\mathscr{L}_c)$, $(\mathscr{S}_c')$ is:

$$\hat{A} := \begin{bmatrix} A & BD \\ -LH & A + LH + BD \end{bmatrix}.$$

Using $T := \begin{bmatrix} I & I \\ 0 & I \end{bmatrix}$, a short computation yields $\sigma(\hat{A}) = \sigma(T^{-1}\hat{A}T) = \sigma(A + BD) \cup \sigma(A + LH)$. Hence $\hat{A}$ is (c)-stable.

Remark. Note that $(\mathscr{S}_c')$ is obtained by applying a direct stabilization to the output of the asymptotic state estimator given in the proof of Theorem 6, iii) $\Rightarrow$ i).

5. *Sampling*

We say that the system $(\mathscr{L}_c)$ is *sampled* if one allows only controls which are constant on the intervals $(k\tau, (k+1)\tau)$ $(k=0, 1, \dots)$, and if from the output y only the values $y(k\tau)$ $(k=0, 1, \dots)$ are assumed to be known. Here τ is some positive number. Therefore, by sampling we obtain the following discrete system:

$$(\mathscr{L}_{cs}): \qquad \bar{x}(\theta+1)=e^{\tau A}\bar{x}(\theta)+\gamma(A)\,B\bar{u}(\theta),\ \ \bar{y}(\theta)=H\bar{x}(\theta),$$

for $\theta\in T_d$. Here $\theta:=t/\tau$, $\bar{x}(\theta):=x(t)$, $\bar{y}(\theta):=y(t)$, $\bar{u}(\theta):=u(t)$, and

$$\gamma(z):=\int_0^\tau e^{\zeta z}\,d\zeta.$$

We will call system $(\mathscr{L}_c)$ *properly sampled* if the condition

$$\lambda\not\equiv\mu \pmod{2\pi i\tau} \qquad (\lambda,\mu\in\sigma(A))$$

is satisfied. It is shown in [1] that, if $(\mathscr{L}_c)$ is properly sampled, the system $(\mathscr{L}_{cs})$ is controllable if and only if $(\mathscr{L}_c)$ is controllable, and $(\mathscr{L}_{cs})$ is observable if and only if $(\mathscr{L}_c)$ is observable. Actually it is proved there that $e^{\tau\lambda}$ is $(e^{\tau A}, B)$-controllable if and only if λ is (A, B)-controllable, and $e^{\tau\lambda}$ is $(e^{\tau A}, H)$-observable if and only if λ is (A, H)-observable. From this observation and from the results of the previous sections it is easily shown that we have the following result:

Theorem 8. Let $(\mathscr{L}_c)$ be properly sampled, then we have:

i) $(\mathscr{L}_{cs})$ is (state-)stabilizable if and only if $(\mathscr{L}_c)$ is (state-)stabilizable.

ii) $(\mathscr{L}_{cs})$ is indirectly (output-)stabilizable if and only if $(\mathscr{L}_c)$ is indirectly (output-)stabilizable.

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